Nature of Games

Game theory is the logical analysis of situations of conflict and cooperation, which consist of the following attributes

1. There are at least two players
   - Players can be individuals, companies, countries, ...
2. Each player has strategies, courses of action, that they can follow
3. Strategies chosen by each player determine the outcome of the game
4. Each outcome is associated with a collection of numerical payoffs

Game theory is the study of how players should rationally play games, where they play to achieve the best possible outcome
Game Theory

- Type of applied mathematics used in social sciences and economics
  - Model behavior in strategic situations, in which an individual’s success in making choices depends on the choices of others
  - Three mathematical forms: extensive, normal, and characteristic

- Extensive form represents the game tree
  - Each node is a state of play, terminal nodes represent payoffs

Two players $A$ and $B$, both put one dollar in the ante. Player $A$ is dealt a card facedown and can bet or check. If player $A$ checks, the the card is inspected; if its a winning card player wins ante, otherwise player $A$ loses. If player $A$ bets, player $A$ must put 2 more dollars in the ante. Then player $B$ not knowing the card must fold or call. If player $B$ folds they lose the ante, if player $B$ calls then player $B$ must put 2 dollars in the ante. Card is then inspected and winner is determined. (diagram is with respect to player $A$)

- Normal form is typically represented by a matrix
  - Matrix shows the players, strategies, and payoffs

<table>
<thead>
<tr>
<th>Player $A$</th>
<th>Player $B$</th>
</tr>
</thead>
<tbody>
<tr>
<td>up</td>
<td>left</td>
</tr>
<tr>
<td></td>
<td>(5, 3)</td>
</tr>
<tr>
<td></td>
<td>right</td>
</tr>
<tr>
<td></td>
<td>(−1, −1)</td>
</tr>
<tr>
<td>down</td>
<td>left</td>
</tr>
<tr>
<td></td>
<td>(0, 0)</td>
</tr>
<tr>
<td></td>
<td>right</td>
</tr>
<tr>
<td></td>
<td>(3, 4)</td>
</tr>
</tbody>
</table>

For example, player $A$ can move up or down while player $B$ can move left or right. If player $A$ moves up and player $B$ moves left, then player $A$ gets payoff of 5 and player $B$ gets payoff of 3...

- General formulation requires the game to be defined using sets
Normal or Strategic Form

- A two-person zero-sum game can be defined by the triplet \((X, Y, F)\)
  - \(X\) is a non-empty set, set of strategies of player \(A\)
  - \(Y\) is a non-empty set, set of strategies of player \(B\)
  - \(F\) is a real-valued function defined on \(X \times Y\), thus \(F(x, y)\) is a real number for every \(x \in X\) and \(y \in Y\)

- Interpretation is as follows
  - Simultaneously player \(A\) chooses \(x \in X\) and player \(B\) chooses \(y \in Y\)
  - \(F(x, y)\) is the resulting reward given to player \(A\) (if negative...)

- Although a simple definition, it can describe any finite combination game such as tic-tac-toe, chess, ...

Are You Even or Odd?

- Two players, \(A\) and \(B\), simultaneously call out the number 1 or 2
  - If the resulting sum is odd, player \(A\) wins; otherwise player \(B\) wins
  - Amount given to the player is the sum of the numbers

- Therefore \(X = \{1, 2\}, Y = \{1, 2\},\) and

<table>
<thead>
<tr>
<th>Player</th>
<th>1</th>
<th>2</th>
</tr>
</thead>
<tbody>
<tr>
<td>(A)</td>
<td>(-2, 2)</td>
<td>(3, -3)</td>
</tr>
<tr>
<td>(B)</td>
<td>(3, -3)</td>
<td>(-4, 4)</td>
</tr>
</tbody>
</table>

NB, the 2 payoffs for each outcome add to zero, therefore this game is considered a **zero-sum game**

*Is there a single strategy each player should select?*
<table>
<thead>
<tr>
<th></th>
<th>Player A</th>
<th>Player B</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>1</td>
<td>2</td>
</tr>
<tr>
<td>1</td>
<td>-2</td>
<td>+3</td>
</tr>
<tr>
<td>2</td>
<td>+3</td>
<td>-4</td>
</tr>
</tbody>
</table>

- Player A *(consider rows)* can win no more than 3 for either strategy, but only risks 2 with strategy ‘1’, so they should perhaps choose ‘1’

- Player B *(consider cols)* wins 4 with strategy ‘2’, but if they expect player A to play ‘1’, then they lose 3, so they should select strategy ‘1’

- Player A may think that player B will select ‘1’ (because of the above argument), so they then decide to play ‘2’

- Actually there is no single set of strategies (saddle point) for this game

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**Consider Player A Strategies**

- Suppose player A calls ‘1’ $\frac{3}{5}$ of the time and ‘2’ $\frac{2}{5}$ of the time
  - If player B calls ‘1’, player A loses 2 dollars $\frac{3}{5}$ of the time and wins 3 dollars $\frac{2}{5}$ of the time, so on the average player A breaks even

$$-2 \cdot \frac{3}{5} + 3 \cdot \frac{2}{5} = 0$$

  - If player B calls ‘2’, player A wins 3 dollars $\frac{3}{5}$ of the time and loses 4 dollars $\frac{2}{5}$ of the time, so on the average player A wins

$$3 \cdot \frac{3}{5} - 4 \cdot \frac{2}{5} = \frac{1}{5}$$

- Therefore if player A is guaranteed to at least break even no matter what player B does using this mixed strategy

*Can player A play the game such that they always win?*
Equalizing Strategy

- Let $p$ denote the proportion of times player $A$ calls ‘1’
  - Choose $p$ so that player $A$ always wins the same amount

<table>
<thead>
<tr>
<th>Player $A$</th>
<th>Player $B$</th>
<th>player B plays ‘1’</th>
<th>player B plays ‘2’</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>1</td>
<td>$-2p + 3(1-p)$</td>
<td>$3p - 4(1-p)$</td>
</tr>
<tr>
<td>2</td>
<td>2</td>
<td>$3 - 5p = 7p - 4$</td>
<td>$12p = 7$</td>
</tr>
<tr>
<td></td>
<td></td>
<td>$p = \frac{7}{12}$</td>
<td></td>
</tr>
</tbody>
</table>

- Therefore player $A$ should call ‘1’ with probability $\frac{7}{12}$, otherwise ‘2’
  - Player $A$ will win on average $-2\frac{7}{12} + 3\frac{5}{12} = \frac{1}{12}$
  - Called the **equalizing strategy** for player $A$

*Player 1 has done some fancy math. What should player 2 do?*

- Player $B$ can do the same process, but will result in the same probabilities (**not always the case, but it happens for this example**)

A Plays Equalizing

![Graph showing average game result vs. play for different strategies](image-url)
Player $A$ is a Winner (with or without a yellow shirt)

- The previous game is in favour of player $A$
  - Can do better than $\frac{1}{12}$ per game if player $B$ does not play properly
  - If player $B$ plays optimally, then lose $\frac{1}{12}$ on average, but no worse
  - Therefore $\frac{1}{12}$ is the value of the game
- The procedure that produces the value is called the optimal strategy or a minimax strategy

Pure and Mixed

- There are pure and mixed strategies
  - Refer to the elements of $X$ and $Y$ as pure strategies
  - Choosing at random among pure strategies is a mixed strategy
- In the previous game, player $A$ used a mixed strategy
  - Also assumed the player only cares about the long term average
  - For example, the player would be indifferent between $5$ million dollars guaranteed, versus flipping a coin and receiving $10$ million with probability $\frac{1}{2}$ and nothing with probability $\frac{1}{2}$
- Utility theory is a better basis for the expected payoff
  - Premise is that a player’s value of money is not linear (it’s what makes game shows interesting after all...)
  - A utility function maps happiness/satisfaction to an outcome
Minimax

• A two-person zero-sum game \((X, Y, F)\) is finite if \(X\) and \(Y\) are finite
  – Therefore the even-odd game was finite

• Fundamental theorem of game theory for zero-sum finite games
  – There is a number \(v\), called the value of the game
  – There is a mixed strategy for player \(A\) such that \(A\)'s average gain
    is at least \(v\) no matter what \(B\) does, and
  – There is a mixed strategy for player \(B\) such that \(B\)'s average loss is
    at most \(v\) no matter what \(A\) does
  – Thanks von Neumann...

  What is the implication if \(v\) is zero, positive, or negative?

Matrix Games

• Finite two-person zero-sum game in a strategic form is a matrix game
  – Payoff function \(F\) can be represented as a matrix
    \[
    F = \begin{pmatrix}
    a_{1,1} & \cdots & a_{1,n} \\
    \vdots & \ddots & \vdots \\
    a_{m,1} & \cdots & a_{n,m}
    \end{pmatrix}
    \]
    where \(a_{i,j} = F(x_i, y_j)\)
  – In this form player \(A\) chooses a row, player \(B\) chooses a column

• Mixed strategy for player \(A\) can be represented as a \(m\)-tuple,
  \(p = (p_1, p_2, \ldots, p_m)\) of probabilities that sum to 1

• Mixed strategy for player \(B\) can be represented as a \(n\)-tuple,
  \(q = (q_1, q_2, \ldots, q_m)\) of probabilities that sum to 1
• Using the matrix, the payoffs can be determined
  – If player A uses mixed strategy \( p \) and player B chooses column \( j \) (pure strategy) then payoff to A is
    \[
    \sum_{i=1}^{m} p_i a_{i,j}
    \]
  – If player A uses \( p \) and player B uses \( q \) then the payoff to A is
    \[
    p^T A q = \sum_{i=1}^{m} \sum_{j=1}^{n} p_i a_{i,j} q_j
    \]

• Using this representation it may be possible to solve the game, which finds the optimal strategy

Saddles, yee haw...

• Occasionally it is easy to solve the game, an entry \( a_{i,j} \) in \( F \) is called a saddle point if it has the following properties
  – \( a_{i,j} \) is the minimum of the \( i^{th} \) row (best for B), and
  – \( a_{i,j} \) is the maximum of the \( j^{th} \) column (best for A)

• At the saddle point player A can win at least \( a_{i,j} \) by choosing row \( i \) and player B can keep the loses to at most \( a_{i,j} \) by choosing column \( j \), this is a pure strategy

• For example,
  \[
  F = \begin{pmatrix} 4 & 1 & -3 \\ 3 & 2 & 5 \\ 0 & 1 & 6 \end{pmatrix}
  \]
  – The entry \( a_{1,1} = 2 \) is the saddle point
  – Value of game is 2, and \((0,1,0)\) is optimal mixed strategy for both
- For large \( m \times n \) matrices it can be tedious to check entries
  - Easier to compute min of each row and max of each column, then check if they match

Example 1

\[
\begin{array}{ccc}
3 & 2 & 1 & 0 \\
0 & 1 & 2 & 0 \\
1 & 0 & 2 & 1 \\
3 & 1 & 2 & 2 \\
\end{array}
\]

row min: 0

\[
\begin{array}{ccc}
3 & 1 & 1 & 0 \\
0 & 1 & 2 & 0 \\
1 & 0 & 2 & 1 \\
3 & 1 & 2 & 2 \\
\end{array}
\]

row min: 0

\[
\begin{array}{ccc}
3 & 2 & 2 & 2 \\
0 & 1 & 2 & 1 \\
1 & 0 & 2 & 1 \\
3 & 1 & 2 & 2 \\
\end{array}
\]

col max: 3

Example 2

- In the first example, no row minimum is equal to any column maximum, so no saddle point
- In the second example, the minimum of the fourth row is equal to the maximum of the second column, so \( a_{3,1} = 1 \) is the saddle point

(assume start counting rows and columns at zero...)
The Solution to $2 \times 2$ Matrix Games

• To solve for the general $2 \times 2$ matrix games, try
  – Test for saddle point (playing pure strategies)
  – If no saddle point, solve by finding equalization strategies

• But many games (more interesting ones) are larger than $2 \times 2$
  – Sometimes these can be reduced to an equivalent $2 \times 2$ game
  – Delete rows and columns which are obviously bad for a player

**Dominates:** Row $i$ of $F$ dominates row $k$ if $a_{i,j} \geq a_{k,j}, \forall j$. Row $i$ strictly dominates row $k$ if $a_{i,j} > a_{k,j}, \forall j$. Similarly, the column $j$ of $A$ dominates column $k$ if $a_{i,j} \leq a_{i,k}, \forall i$ (strictly $a_{i,j} < a_{i,k}$).

• Anything that player $A$ can achieve using a dominated row can be achieve at least as well using the row it dominates, therefore dominated rows can be removed. Similar argument for player $B$.

For example,

\[
F = \begin{pmatrix}
2 & 0 & 4 \\
1 & 2 & 3 \\
4 & 1 & 2 
\end{pmatrix}
\]

• Last column is dominated by the middle so remove, giving

\[
F = \begin{pmatrix}
2 & 0 \\
1 & 2 \\
4 & 1 
\end{pmatrix}
\]

• Top row is dominated by the bottom row so remove, giving

\[
F = \begin{pmatrix}
1 & 2 \\
4 & 1 
\end{pmatrix}
\]
Other Complications

- Consider outcomes that do not equal zero

<table>
<thead>
<tr>
<th></th>
<th>Player B</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>1</td>
</tr>
<tr>
<td>1</td>
<td>(1, 1)</td>
</tr>
<tr>
<td>2</td>
<td>(2, -2)</td>
</tr>
</tbody>
</table>

- Most attractive outcome seems to be (1, 1), but both players may try to take advantage of the situation and end up with (-5, -5)

- Player may be able to communicate about strategies

- There may be more than 2 players, and coalitions may be created